A Conceptually Simple Proof of the Quantum Reverse Shannon Theorem

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Abstract. The Quantum Reverse Shannon Theorem states that any quantum channel can be simulated by an unlimited amount of shared entanglement and an amount of classical communication equal to the channel's entanglement assisted classical capacity. In this paper, we provide a new and conceptually simple proof of this theorem, which has previously been proved in [Bennett et al., arXiv:0912.5537]. Our proof is based on optimal one-shot Quantum State Merging and the Post-Selection Technique for quantum channels.

In 1948 Shannon derived his famous 'Noisy Channel Coding Theorem' [1], which shows that the capacity C of a classical channel $\mathcal{J}$ is given by the maximum, over the input distributions X, of the input/output mutual information $C = \max_X (H(X) + H(\mathcal{J}(X)) - H(X, \mathcal{J}(X)))$. Shannon also showed that the capacity does not increase if one allows to use shared randomness between the sender and the receiver. In 2001 Bennett et al. [2] proved the 'Classical Reverse Shannon Theorem' which states that, given free shared randomness between the sender and the receiver, every channel can be simulated using an amount of classical communication equal to the capacity of the channel. This implies that in the presence of free shared randomness, the capacity of a channel $\mathcal{J}$ to simulate another channel $\mathcal{I}$ is given by $C_R(\mathcal{J}, \mathcal{I}) = \frac{C(\mathcal{J})}{C(\mathcal{I})}$ and hence only a single parameter remains to characterize classical channels.

In contrast to the classical case, a quantum channel has various distinct capacities. In [2] Bennett et al. argue that the entanglement assisted classical capacity C_E of a quantum channel $\mathcal{E}$ is the natural quantum generalization of the classical capacity of a classical channel. They show that the entanglement assisted classical capacity is given by the quantum mutual information $C_E = \max_\rho (H(\rho) + H(\mathcal{E}(\rho)) - H((\mathcal{E} \otimes \mathrm{id})\Phi_\rho))$, where the maximum goes over all input distributions ρ and Φ_ρ is a purification of ρ. Motivated by this, they conjectured the 'Quantum Reverse Shannon Theorem (QRST)' [2], which Bennett et. al subsequently proved in [3]. The theorem states that any quantum channel can be simulated by an unlimited amount of shared entanglement and an amount of classical communication equal to the channel's entanglement assisted classical capacity. So if entanglement is for free, the capacity of a quantum channel $\mathcal{E}$ to simulate another quantum channel $\mathcal{F}$ is given by $C_E(\mathcal{E},\mathcal{F}) = \frac{C_E(\mathcal{E})}{C_E(\mathcal{F})}$ and hence only a single parameter remains to characterize quantum channels.

For the QRST it surprisingly turned out that maximally entangled states are not the appropriate resource for general input sources. This is because of an issue known as entanglement spread [4]. If we change the entanglement resource to embezzling states however, this problem can be overcome. The embezzling states [5] have the form $|\mu(k)\rangle_{AB} \propto \sum_{j=1}^{k} \frac{1}{\sqrt{j}} |jj\rangle_{AB}$ and the transformation $|\mu(k)\rangle_{AB} \mapsto |\mu(k)\rangle_{AB} \otimes |\varphi\rangle_{AB}$ can be accomplished with fidelity better than $(1-\epsilon)$ for $k > m^{1/\epsilon}$ without communication for any $|\varphi\rangle_{AB}$ with Schmidt rank m.

Here we present a new proof of the QRST that is based on a new one-shot version for 'Quantum State Merging/State Splitting' [6, 7] and the 'Post-Selection Technique' [8] for quantum channels. The idea is as follows. Let $\mathcal{E}$ be a quantum channel that takes inputs ρ_A on Alice's side and outputs $\mathcal{E}(\rho_A)$ on Bob's side. We like to think of $\mathcal{E}$ as $\mathcal{E}(\rho_A) = (\mathrm{id}_{A\to B})\mathrm{tr}_{A'}(U_{AA'}(\rho_A \otimes |0\rangle\langle 0|_{A'})U_{AA'}^{\dagger})$, where A' is an additional register at Alice's side and $U_{AA'}$ is some unitary (Stinespring dilation). Now we first simulate the quantum channel locally at Alice's side, giving her $\sigma_{AA'} = U_{AA'}(\rho_A \otimes |0\rangle\langle 0|_{A'})U_{AA'}^{\dagger}$, and then use State Splitting to do an optimal state transfer of the A-part to Bob's side, such that he holds $\mathcal{E}(\rho_A) = (\mathrm{id}_{A\to B})\sigma_A$ in the end. This simulates the channel $\mathcal{E}$ and using the Post-Selection Technique we can show that the asymptotic classical communication rate of the State Splitting protocol is given by C_E.

References

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