The maximally entangled symmetric state in terms of the geometric measure

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Abstract. The maximally entangled permutation-symmetric state in terms of the geometric measure of entanglement is investigated for pure states of multipartite systems. We focus on the n qubit case, putting forward candidates for maximal symmetric entanglement for up to twelve qubits. These states can be visualized by points on a sphere via the so-called Majorana Representation, thus yielding valuable information about the nature of the states. Strong lower bounds on the entanglement are readily provided by known solutions of classical optimization problems on the S^2 sphere, namely Tóth's problem and Thomson's problem.

1 Introduction

Multipartite entanglement is a fundamental resource in a wide range of situations in quantum information processing. Here we investigate the geometric measure of entanglement [1, 2] for states of n qubits. This entanglement measure is defined as the maximal overlap of a given pure state $|\psi\rangle$ with all pure product states:

$$E_G(|\psi\rangle) = \min_{|\lambda\rangle \in \mathcal{H}_{\text{SEP}}} - \log_2 |\langle \lambda | \psi \rangle|^2 = - \log_2 |\langle \Lambda | \psi \rangle|^2 \ . \qquad (1)$$

E_G has close links to other distance-like entanglement measures, such as the robustness of entanglement and the relative entropy of entanglement.

Symmetric quantum states of n qubits can be written as $|\psi\rangle_{\text{s}} = \sum_{k=0}^{n} a_k |S_k\rangle$, where the $|S_k\rangle$ are the orthonormalized symmetric basis states. Symmetric states play a crucial role in many experiments, and they are candidates for MBQC. Most quantum states are too entangled for being computationally universal [3], so that determining the comparatively low entanglement of n qubit symmetric states (maximal entanglement between $\mathcal{O}(\log \sqrt{n})$ and $\mathcal{O}(\log n)$) could be valuable for MBQC.

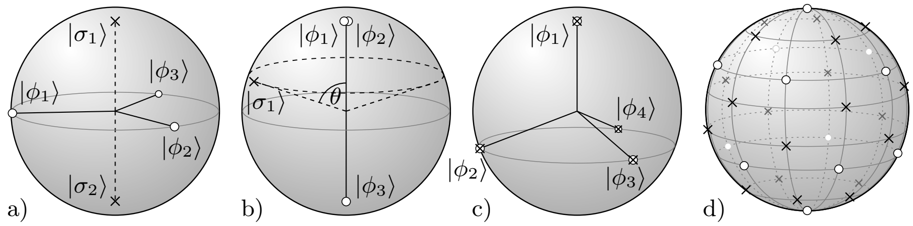

Fig. 1. Majorana Representation of some highly or maximally entangled symmetric states, with the MPs depicted as white circles and the CPPs as crosses. a) shows the three qubit GHZ state $|000\rangle + |111\rangle = |S_0\rangle + |S_3\rangle$, b) the three qubit W state $|S_1\rangle$, c) the four qubit "tetrahedron state" $|S_0\rangle + \sqrt{2}|S_3\rangle$, and d) the 12 qubit "icosahedron state" $\sqrt{7}|S_1\rangle - \sqrt{11}|S_6\rangle - \sqrt{7}|S_{11}\rangle$. All of these states have more than one CPP.

2 Results

By means of the Majorana Representation [4] every symmetric state of n qubits $|\psi\rangle_{\mathrm{s}}$ can be unambiguously associated with n single qubits:

$$|\psi\rangle_{\mathrm{s}} = \frac{1}{\sqrt{K}} \sum_{\mathrm{perm}} |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_n\rangle \ . \tag{2}$$

The qubits $|\phi_i\rangle$ are uniquely determined by the choice of $|\psi\rangle_{\mathrm{s}}$ and they determine the normalization factor K. We call the $|\phi_i\rangle$ the *Majorana points* (MP) of $|\psi\rangle_{\mathrm{s}}$. For symmetric states the closest product state $|\Lambda\rangle$ from Eq. (1) is always symmetric itself [5]. Therefore we have $|\Lambda\rangle = |\sigma\rangle^{\otimes n}$, and we call the single qubit state $|\sigma\rangle$ a *closest product point* (CPP) of $|\psi\rangle_{\mathrm{s}}$.

A given $|\psi\rangle_{\mathrm{s}}$ can be visualized on a sphere by displaying the Bloch vectors of its MPs and CPPs, as seen in Figure 1. For up to 12 qubits we numerically determined the maximally entangled states among symmetric states with positive coefficients, and found candidates for the general symmetric case. This was aided by the aforementioned visualization.

Our optimization problem on the sphere is related to the classical problems of Tóth (maximize the minimum pairwise distance of n points) and Thomson (minimum energy configuration of n electrons confined to the surface of a sphere). We use their known solutions to compute powerful lower bounds on the maximal symmetric entanglement.

References

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